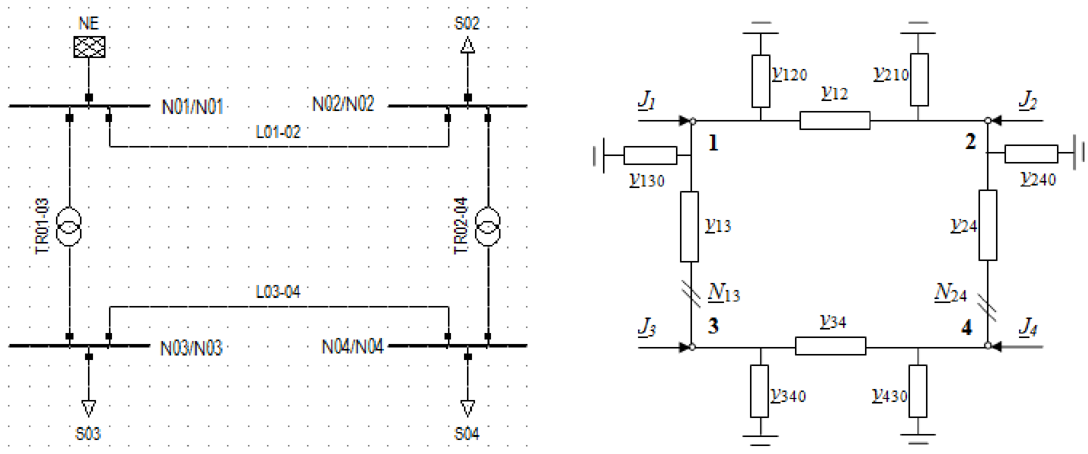
C06 - The bus admittance matrix and non-linear load flow models

The first stage in a load flow analysis is the building of the one-line diagram of the studied EPS, for which the elements' electrical parameters are defined, and then the system loading, expressed through the active and reactive consumed and generated power measured at its buses, is needed.

Load flow algorithms use subsequently the electrical parameters to build equivalent mathematical representations for all EPS elements (for instance, Γ equivalent circuits are used for transformers, Π equivalent circuits are used for lines, and constant values are used for loads).

Thus, for the one-line diagram depicted in Fig. 6.1 (a), the equivalent one-line electrical representation which uses Γ and Π circuits is the one in Fig. 6.1 (b).



(a) (b)

Fig. 6.1 – The one line diagram and equivalent representation of a simple EPS

In the diagram in Fig. 6.1 (a), loads are represented as currents $\underline{J}_i$, and the transformer ratios are complex numbers. The common ground node 0 is the reference bus, and buses 1-4 are independent buses.

Next, the bus shunt admittances are computed as sums of shunt admittances of EPS elements connected to the bus (eq. 6.1). Applying this reduction, the equivalent electrical circuit will have only 4 series and 4 shunt branches, as seen in Fig. 6.2. In the circuit in Fig. 6.2, the following assumptions are made:

$$
\begin{aligned}
\underline{y}_{10} &= \underline{y}_{120} + \underline{y}_{130} \\
\underline{y}_{20} &= \underline{y}_{210} + \underline{y}_{240} \\
\underline{y}_{30} &= \underline{y}_{340} \\
\underline{y}_{40} &= \underline{y}_{430}
\end{aligned}
\tag{6.1}
$$

- on the series branches, currents flow from the smaller number order bus to the bigger number order bus, and voltages directions follow the direction of the current
- on the shunt branches, currents flow to ground and voltages are directed towards the neutral (ground) bus.
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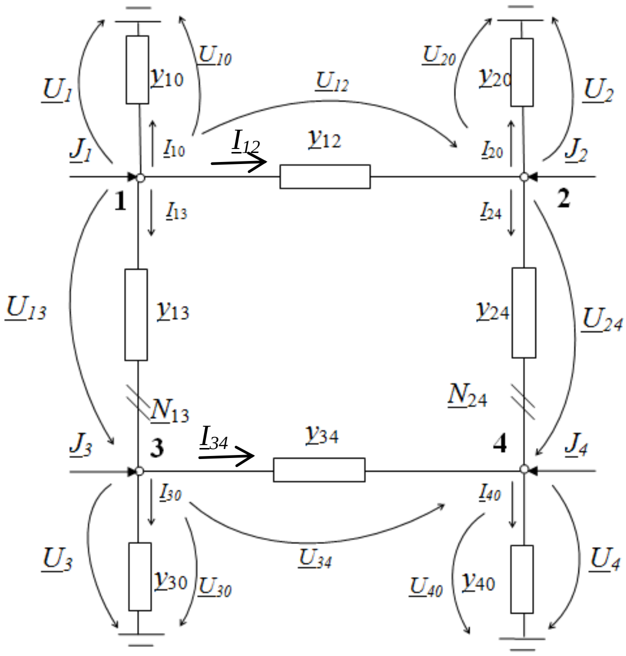

Fig. 6.2 – The final equivalent representation of the simple EPS

The bus current and voltage vectors are written as:

$$
[\underline{J}_n] = \begin{bmatrix} \underline{J}_1 \\ \underline{J}_2 \\ \underline{J}_3 \\ \underline{J}_4 \end{bmatrix} \qquad [\underline{U}_n] = \begin{bmatrix} \underline{U}_1 \\ \underline{U}_2 \\ \underline{U}_3 \\ \underline{U}_4 \end{bmatrix} \tag{6.2}
$$

For the circuit in Fig. 6.2, an incidence graph is attached, in which branches are oriented according to the currents flow (Fig. 6.3). If branches are numbered as in Fig. 6.3, branch voltage drops and branch currents can be written in vector form as:

$$
\begin{aligned}
[\underline{U}] &= [\underline{U}_{10}, \underline{U}_{20}, \underline{U}_{30}, \underline{U}_{40}, \underline{U}_{12}, \underline{U}_{13}, \underline{U}_{23}, \underline{U}_{34}]^T \\
[\underline{I}] &= [\underline{I}_{10}, \underline{I}_{20}, \underline{I}_{30}, \underline{I}_{40}, \underline{I}_{12}, \underline{I}_{13}, \underline{I}_{23}, \underline{I}_{34}]^T
\end{aligned}
\tag{6.3}
$$

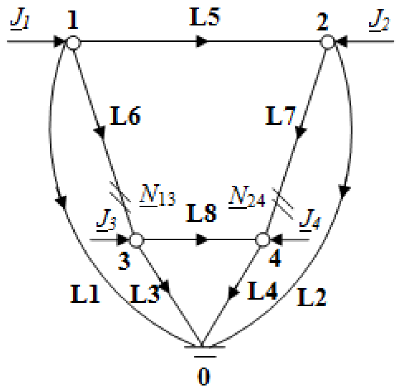

Fig. 6.3 – The EPS graph

Using these notations, two topological equations (which involve state variables, i.e. voltages, which depend on EPS connectivity) can be written.

If the incidence matrix, the matrix that encodes the connectivity between buses and branches is denoted by [A], the following equations can be written:

$$[\underline{A}]^* \cdot [\underline{I}] = [\underline{J}_n] \qquad (6.4),$$

which is the first Kirchhoff law, describing the relation between branch and bus currents, and

$$[\underline{A}]^T \cdot [\underline{U}_n] = [\underline{U}] \qquad (6.5)$$

the connection between bus voltages and branch voltage drops.

Matrix [A] has a number of rows equal to the number of independent buses in the EPS, and a number of columns equal to the number of EPS branches. Its elements, denoted by a_{ij} (where i is the row index, and j is the column index) can have the following values:

- 0, if branch j does not connect in bus i
- +1 or -1 if branch j is directly connected in bus i, in positive or negative direction
- $+\underline{N}_{ki}$ sau $-\underline{N}_{ki}$ if branch j is connected with bus i through the transformer ratio side of a transformer, in positive or negative direction, and directly (via impedance) connected to bus *k*.

For the EPS graph in Fig. 3, the [A] matrix is written as:

$$[\underline{A}] = \begin{array}{c|cccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ & \text{L1-0} & \text{L2-0} & \text{L3-0} & \text{L4-0} & \text{L1-2} & \text{L1-3} & \text{L2-4} & \text{L3-4} \\ \hline \text{N1} & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ \text{N2} & 0 & 1 & 0 & 0 & -1 & 0 & 1 & 0 \\ \text{N3} & 0 & 0 & 1 & 0 & 0 & -\underline{N}_{13} & 0 & 1 \\ \text{N4} & 0 & 0 & 0 & 1 & 0 & 0 & -\underline{N}_{24} & -1 \end{array}$$

A second type of equations that can be written for an EPS are electrical equations, which use also electrical parameters, in this case admittances.

On each branch, the voltage drop can be computed using the branch admittance and branch current. In matrix form, for the entire system, this can be written as:

$$[\underline{I}] = \lfloor \underline{y} \rfloor \cdot [\underline{U}] \tag{6.6}$$

where

$$\lfloor \underline{y} \rfloor = diag(\underline{y}_{10}, \underline{y}_{20}, \underline{y}_{30}, \underline{y}_{40}, \underline{y}_{12}, \underline{y}_{13}, \underline{y}_{24}, \underline{y}_{34}) \tag{6.7}$$

If we substitute $[\underline{I}]$ using (6.7) and $[\underline{U}]$ using (6.5) in (6.4), we get:

$$[\underline{A}]^* \cdot \lfloor \underline{y} \rfloor \cdot [\underline{A}]^T \cdot [\underline{U}_n] = [\underline{J}_n] \tag{6.8}$$

If we denote in (6.8)

$$[\underline{A}]^* \cdot \lfloor \underline{y} \rfloor \cdot [\underline{A}]^T \equiv [\underline{Y}_n], \tag{6.9}$$

equation (6.8) can be rewritten as

$$[\underline{Y}_n] \cdot [\underline{U}_n] = [\underline{J}_n] \tag{6.10}$$

that gives the general linear load flow model, in which $[\underline{U}_n]$ is the unknown or state variables vector, to be computed using $[\underline{J}_n]$ the load and $[\underline{Y}_n]$, the bus admittance matrix.

The bus admittance matrix $[\underline{Y}_n]$ is a square matrix, with a number of rows equal to the number of EPS independent buses, and describes electrically and formally the EPS connectivity. Matrix $[\underline{Y}_n]$ can be computed with (6.9) or, alternately, in the following manner:

(1) If the analyzed EPS does not have transformers, $[\underline{Y}_n]$ is a symmetrical matrix and its elements are computed as follows:

- its main diagonal elements, $\underline{Y}_{ii}$, have their value equal to the sum of branch admittances connected to bus *i*
- the elements outside the main diagonal, $\underline{Y}_{ij}$, have the value equal to the admittance of the branch connecting buses *i* and *j*, with minus sign. If no direct branch connection exists between buses *i* and *j*, this value is equal to 0.

(2) If the EPS has ideal transformers:

- the main diagonal elements, $\underline{Y}_{ii}$, are computed as the sum of elements connected directly, with impedance, to bus i, and the sum of admittances connected through a ideal transformer ratio times the square of the transformer ratio:

$$\underline{Y}_{ii} = \sum_{k \in L_i} N_{ki}^2 \cdot \underline{y}_k \tag{6.11}$$

where L_i denotes the set of branches connected in bus i and N_{ki} is the ratio of the transformer from branch k connected to bus i. If branch k is connected through impedance with bus i, then $N_{ki}=1$.

- the elements outside the main diagonal, $\underline{Y}_{ij}$, have values equal to the branch connected between buses i and j, with minus sign, times the transformer ratio. If the transformer ratios in the equivalent circuit is at side i, the complex transformer ratio is conjugated.

$$\underline{Y}_{ij} = -N_{ki}^{*} \cdot \underline{y}_k \qquad \underline{Y}_{ji} = -N_{ki} \cdot \underline{y}_k \tag{6.12}$$

If the transformer ratios are complex numbers, the bus admittance matrix $[\underline{Y}_n]$ is no longer symmetrical.

For the EPS in Figures (6.1) - (6.3), the bus admittance matrix is written as:

$$[\underline{Y}_n] = \begin{array}{c|c|c|c|c} & \text{N1} & \text{N2} & \text{N3} & \text{N4} \\ \hline \text{N1} & \underline{y}_{10}+\underline{y}_{12}+\underline{y}_{13} & -\underline{y}_{12} & -\underline{y}_{13}\cdot\underline{N}_{13} & 0 \\ \hline \text{N2} & -\underline{y}_{12} & \underline{y}_{20}+\underline{y}_{12}+\underline{y}_{24} & 0 & -\underline{y}_{24}\cdot\underline{N}_{24} \\ \hline \text{N3} & -\underline{y}_{13}\cdot\underline{N}_{13}^{*} & 0 & \underline{y}_{30}+\underline{y}_{34}+\underline{y}_{13}\cdot\underline{N}_{13}^{2} & -\underline{y}_{34} \\ \hline \text{N4} & 0 & -\underline{y}_{24}\cdot\underline{N}_{24}^{*} & -\underline{y}_{34} & \underline{y}_{40}+\underline{y}_{34}+\underline{y}_{24}\cdot\underline{N}_{24}^{2} \end{array}$$

Using the bus admittance matrix, a general mathematical model for the load flow problem can be written.

In the equivalent electrical one-line circuit of an EPS with n independent buses, where branches are represented by quadrupoles, each bus together with its connections can be represented as in Fig. 6.4.

For such a bus, the (6.10) equation can be rewritten as:

$$\underline{Y}_{ii} \cdot \underline{U}_i - \sum_{\substack{k=1 \\ k\neq i}}^{n} \underline{Y}_{ik} \cdot \underline{U}_k = \underline{J}_i, \quad i = 1 \ldots n \tag{6.13}$$

$$\text{where} \quad \underline{J}_i = \frac{\underline{S}_i^{*}}{\underline{U}_i^{*}} \tag{6.14}$$

From (6.13) and (6.14), the non-linear mathematical bus model is derived in two steps, in (6.15).

- rewrite (6.13) in which the nodal current $\underline{J}_i$ is replaced with the second term from (6.14)
- rewrite separately the slack bus equation, and in the other equations the terms where the slack bus voltage appears are separated

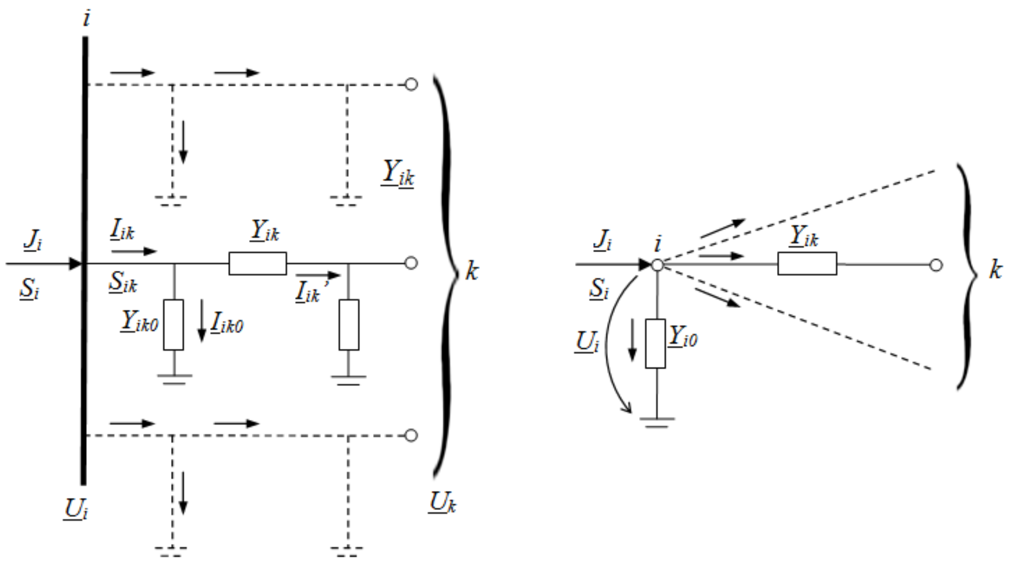

Fig. 6.4 - The equivalent circuit associated to a generic i bus

$$\begin{cases} \underline{Y}_{ii}\cdot\underline{U}_i-\sum\limits_{\substack{k=1\\k\neq i\\k\neq e}}^{n}\underline{Y}_{ik}\cdot\underline{U}_k-\underline{Y}_{ie}\cdot U_e=\dfrac{\underline{S}_i^*}{\underline{U}_i^*} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad ,i\in N_{PU}\cup N_{PQ}, i\neq e \\ \underline{Y}_{ee}\cdot U_e-\sum\limits_{\substack{k=1\\k\neq e}}^{n}\underline{Y}_{ek}\cdot\underline{U}_k=\dfrac{\underline{S}_e^*}{U_e} \end{cases} \tag{6.15}$$

In the (6.15) equations, the slack bus voltage U_e is constant, it has a real value, not complex, and it is considered as angle reference. $N = N_{PU} \cup N_{PQ}$ is the entire set of consumer (PQ) and generator (PV) buses.

The first (6.15) equation can be rewritten in compact, matrix form, as:

$$[\underline{Y}_N]\cdot[\underline{U}_N]=\left[\frac{\underline{S}_N^*}{\underline{U}_N^*}\right]+[\underline{Y}_{Ne}]\cdot U_e \tag{6.16}$$

this writing form being useful in cases when the load flow algorithm processes the whole bus admittance matrix.

After the state variables $[\underline{U}_N]$ are computed from the first (6.15) equation, the second equation can be used in order to compute the active and reactive power injected at the slack bus.

The (6.16) equations system it is not used in this form for large power systems. Alternatively, starting from eq. (6.13), equivalent mathematical models are determined, which express the bus current or power balance. For this purpose, algebraic and trigonometric representations are used for admittances and voltages:

$$\begin{aligned} \underline{Y}_{ii} &= G_{ii}+j\cdot B_{ii} &&= Y_{ii}\cdot e^{j\cdot\Psi_{ii}} &&= Y_{ii}\cdot(\cos\Psi_{ii}+j\cdot\sin\Psi_{ii}) \\ \underline{Y}_{ik} &= G_{ik}+j\cdot B_{ik} &&= Y_{ik}\cdot e^{j\cdot\Psi_{ik}} &&= Y_{ik}\cdot(\cos\Psi_{ik}+j\cdot\sin\Psi_{ik}) \\ \underline{U}_i &= U_i'+j\cdot U_i'' &&= U_i\cdot e^{j\cdot\Theta_i} &&= U_i\cdot(\cos\Theta_i+j\cdot\sin\Theta_i) \end{aligned} \tag{6.17}$$

The bus power balance model

Equations (6.13) and (6.14) can be rewritten as:

$$\underline{S}_i = \underline{U}_i \cdot \underline{J}_i^* = \underline{U}_i \cdot (\underline{Y}_{ii}^* \cdot \underline{U}_i^* - \sum_{\substack{k=1\\k\neq i}}^{n} \underline{Y}_{ik}^* \cdot \underline{U}_k^*) \qquad i = 1...n;\ i \neq e \qquad (6.18)$$

and

$$\underline{Y}_{ii}^* \cdot U_i^2 - \underline{U}_i \cdot \sum_{\substack{k=1\\k\neq i}}^{n} \underline{Y}_{ik}^* \cdot \underline{U}_k^* - (P_i + j \cdot Q_i) = 0 \qquad i = 1...n;\ i \neq e \qquad (6.19)$$

By using the algebraic representation for admittances and the trigonometric representation for voltages:

$$\begin{aligned} &\underline{Y}_{ii} = G_{ii} + j \cdot B_{ii} \qquad \underline{Y}_{ik} = G_{ik} + j \cdot B_{ik} \\ &\underline{U}_i = U_i \cdot e^{j \cdot \Theta_i} = U_i \cdot (\cos\Theta_i + j \cdot \sin\Theta_i) \end{aligned} \qquad (6.20)$$

and separating the real and imaginary parts in (6.19), the following non/linear model is obtained:

$$\begin{aligned} &P_i = G_{ii} \cdot U_i^2 - U_i \cdot \sum_{\substack{k=1\\k\neq i}}^{n} U_k \cdot [G_{ik} \cdot \cos(\theta_i - \theta_k) + B_{ik} \cdot \sin(\theta_i - \theta_k)] \\ &Q_i = -B_{ii} \cdot U_i^2 - U_i \cdot \sum_{\substack{k=1\\k\neq i}}^{n} U_k \cdot [G_{ik} \cdot \sin(\theta_i - \theta_k) - B_{ik} \cdot \cos(\theta_i - \theta_k)] \quad (6.21) \\ &\qquad i = 1...n; \quad i \neq e \end{aligned}$$

This complicated equations system cannot be resolved analytically, numerical methods being used instead. The non-linear expression do not allow the use of direct methods, iterative methods being required.

In order to solve the load flow model, the non-linear equations system, written fundamentally as

$$f(x) = 0 \qquad (6.22)$$

it is solved in the following manner:

- an initial approximation of the sought solution is generated, $\mathbf{x}^{(0)}$, and the iteration counter is set to 1: $t = 1$
- at any step t, with the current approximation $\mathbf{x}^{(t)}$, a new $\mathbf{x}^{(t+1)}$ approximation is computed, which should be better than the previous
- the iterative process continues until a sopping criterion is met

The exact solution it is theoretically found only if an infinite number of iterations is performed. Thus, with a finite number of steps, the found solution will be only an approximation.

The best known iterative load flow algorithms are the Gauss-Seidel method and the Newton-Raphson method.